Matrices

Linear Systems

– Linear equations can be represented as matrix equations

$2x+3y=4$
$-2x+y=8$
$= \begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

Augmented form easier to write:

$= \left(\begin{array}{cc:c} 2 & 3 & 4 \\ -2 & 1 & 8 \end{array}\right)$

Perform Gaussian Elimination to solve

performing operation on one row doesn't affect others directly.

Manipulate rows to find upper triangular form

$\left(\begin{array}{ccc:c} a & b & c & x \\ 0 & d & e & y \\ 0 & 0 & f & z \end{array}\right)$

Solve simultaneously : $f = z$ ... etc.

Can also solve $Ax = B$ with $\boxed{x = A^{-1}B}$ but requires more work

General Cases:

a) $B \neq 0$, $|A| \neq 0$
$x = A^{-1}B$

b) $b = 0$, $|A| \neq 0$
$Ax = 0$
$\therefore x = 0$

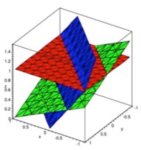
c) $b \neq 0$, $|A| = 0$ *
→ no solutions or ∞

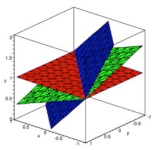
d) $b = 0$, $|A| = 0$
→ ∞ solutions

* if $|A| = 0$, A^{-1} doesn't exist → Singular

∴ solution is not unique or may not exist at all

Geometrical eg.





no solutions (inconsistent) | ∞ solutions (degenerate)

Rank

Why use?

Use rank to determine type of solution:

if rank $(A) \neq$ rank $([A|B])$ ← augmented matrix
→ no solutions

if rank $(A) =$ rank $([A|B])$
→ family of solutions

How to calculate:

Carry out row operations for upper triangular form, then:

$\boxed{\text{Rank}(A) = n - \text{Null}(A)}$

n = number of rows (doesn't have to be square matrix)

Null = number of entirely zero rows:

e.g $\begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$ Null = 1, rank = 3 - 1 = 2

Inverse

$\boxed{\text{Inverse } A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$

Transpose: Rows → Columns

$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \rightarrow \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$

Transpose of matrix of cofactors

Cofactor Matrix

Minor × appropriate sign

take determinant of value in matrix:
e.g for 8

Remember: det. multiplies minor by its value, cofactor just minor × appropriate sign

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \cdots & + & - & + & \cdots \end{pmatrix} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -(6-12) = 6$

cofactor matrix $\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & 6 & \cdots \end{pmatrix}$

appropriate sign : $\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$

Alternative to find A^{-1} using row operations

Augment A with identity matrix and perform row operations to 'switch' them:

e.g $A = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$

$\left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right)$

$A_2 : R_2 + 2A_1$ $\left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$

$R_1 : R_1 - 2R_2$ $\left(\begin{array}{cc:cc} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right)$

$\therefore A^{-1} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$

Determinant

Σ (matrix value × its minor)

for 2x2

$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$

for 3x3

$\boxed{+ \quad - \quad + \quad - \quad \cdots}$

$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$

↳ • take values in any row* in matrix
• 'cross out' row and column the value lies in and multiply value by remaining 2x2 matrix determinant

e.g for $a_{1,2}$

$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \rightarrow a_{1,2}\boxed{\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix}}$

Just use one row

* can use any row for determinant
→ if row has 0's it makes it simpler

Eigenvalues & Eigenvectors

Eigenvalues: scalar (λ) of vector v that gives amount of stretch

Eigenvectors: vectors whose direction is not altered under matrix multiplication

$Av = \lambda v$

$= (A - \lambda I_n)v = 0$

but for " to have a non-trivial solution, the matrix $\boxed{(A - \lambda I_n) \text{ must have full rank}}$

∴ to calculate eigenvalues

$\boxed{\det(A - \lambda I_n) = 0}$

expanding this gives characteristic equation. 2x2 = quadratic, ∴ up to 2 λ's
3x3 = cubic, ∴ up to 3 λ's etc.

To calculate eigenvectors

$\boxed{\text{substitute eigenvalues into } (A - \lambda I_n)v = 0}$

↳ choose unit vector by convention

Useful Rules

eigenvalues of A^{-1} = $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots$

$kA = k\lambda_1, k\lambda_2 \ldots$

trace of matrix is equal to sum of eigenvalues

sum of diagonal entries

$A^k = \lambda_1^k, \lambda_2^k \ldots$

transposed matrix has same eigenvalues

determinant of matrix equals product of eigenvalues